

Breve análisis de algunos métodos de interpolación para series de tiempo de imágenes satelitales

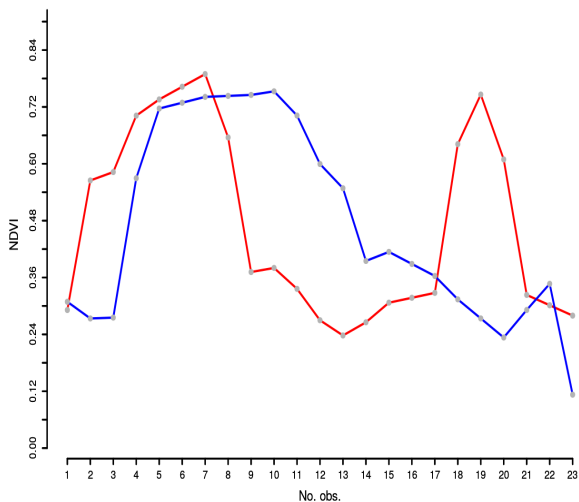
Inder Tecuapetla Gómez¹

¹CONACyT-CONABIO

Taller TATSSI: Primera Parte,
Julio 18, 2019, CONABIO

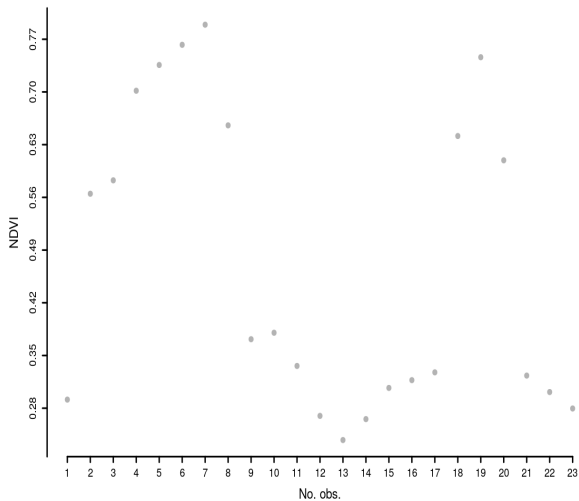


Par de NDVIs

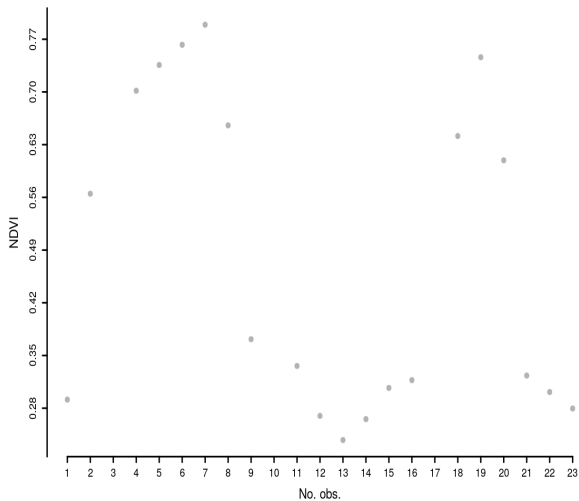


Datos tomados de Colditz et al. (2008)

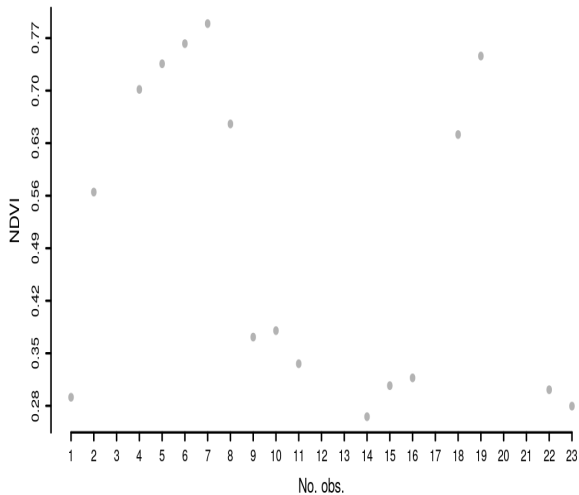
Datos originales



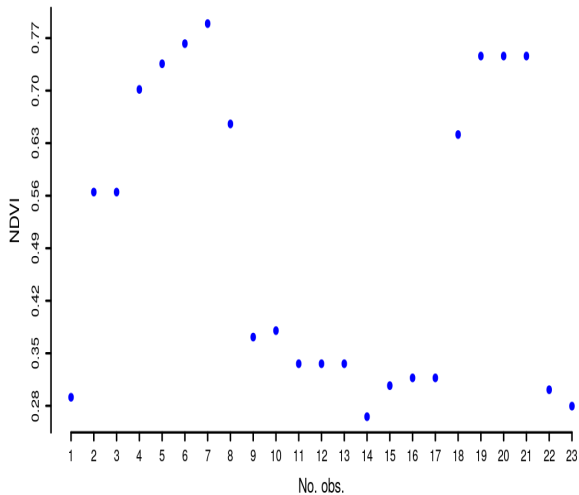
Datos originales



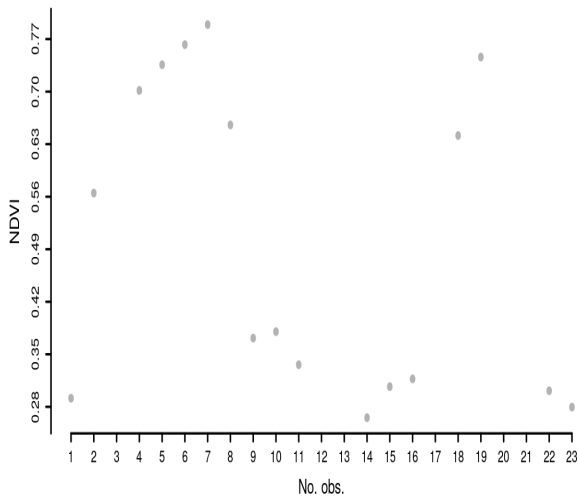
Interpolación Previous



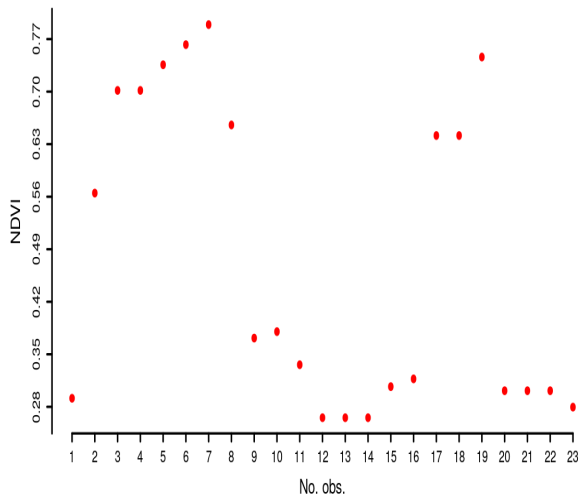
Interpolación Previous



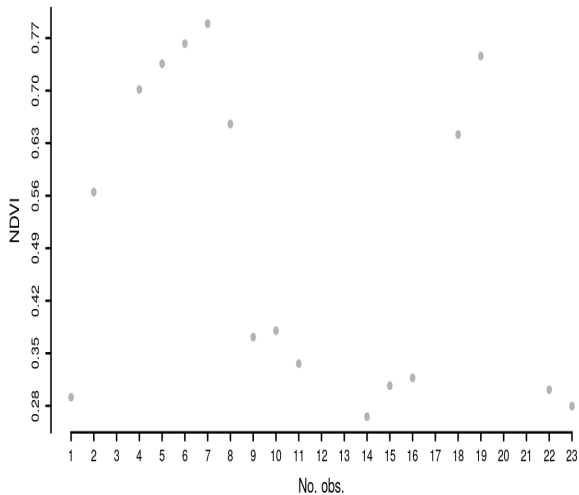
Interpolación Next



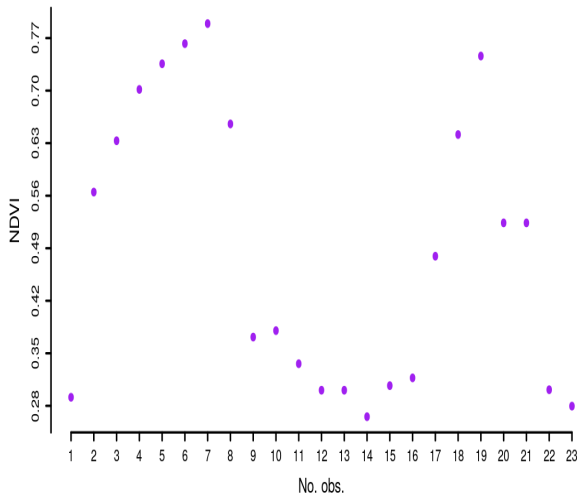
Interpolación Next



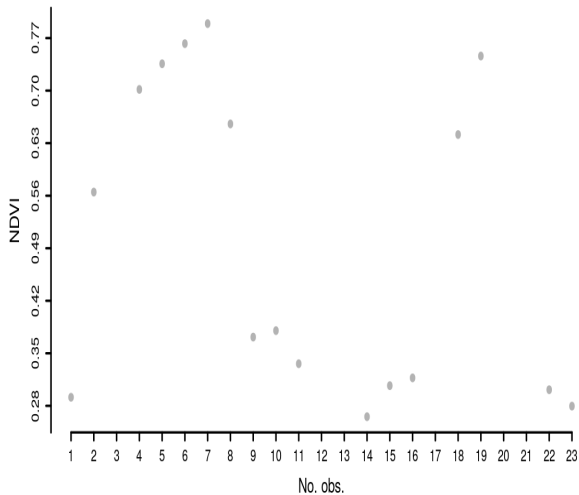
Interpolación Mean



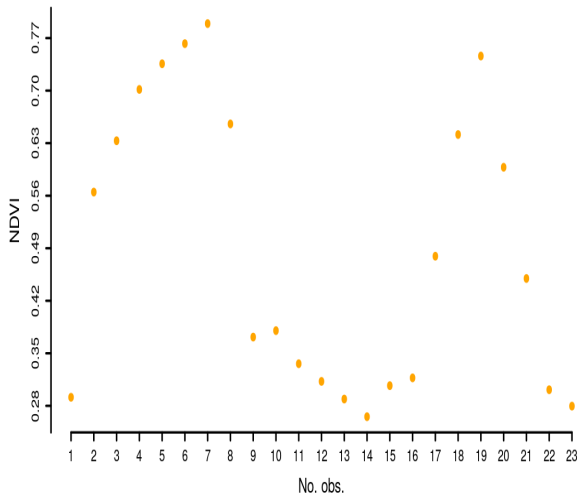
Interpolación Mean



Interpolación Linear



Interpolación Linear



Polinomios de Lagrange

- Se puede probar que dadas las observaciones $(x_1, y_1), \dots, (x_n, y_n)$ el polinomio

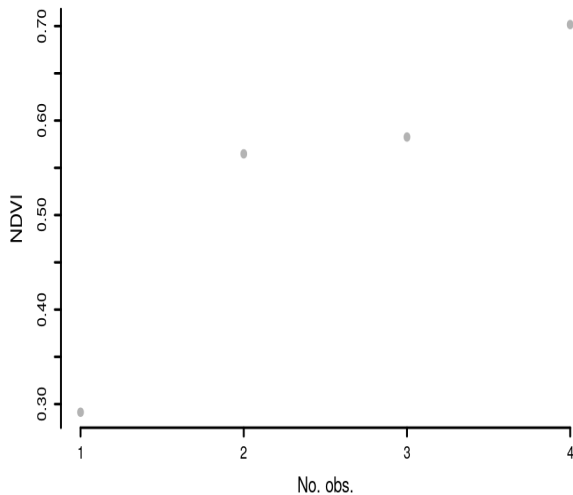
$$p_n(x) = \sum_{j=1}^n c_j(x) y_j,$$

donde

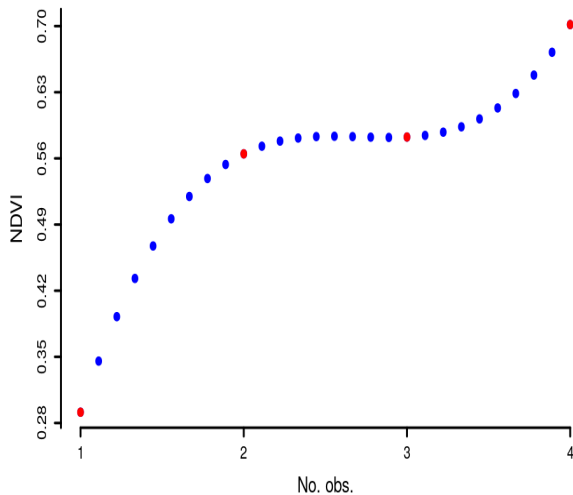
$$c_j(x) = \frac{x - x_1}{x_j - x_1} \dots \frac{x - x_{j-1}}{x_j - x_{j-1}} \frac{x - x_{j+1}}{x_j - x_{j+1}} \dots \frac{x - x_n}{x_j - x_n}$$

describe una gráfica que **pasará** por los valores y_1, y_2, \dots, y_n .

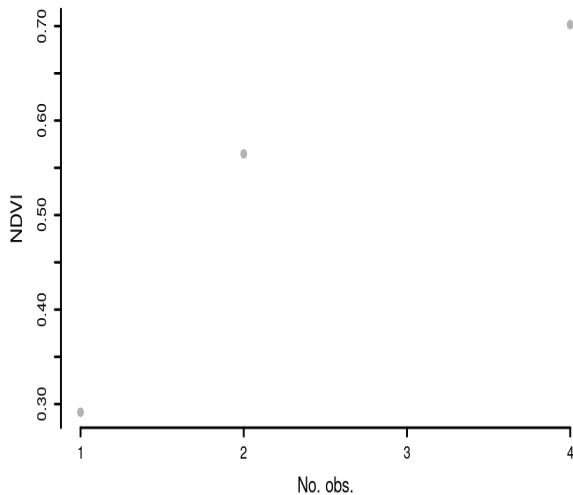
Interpolación Polinomial



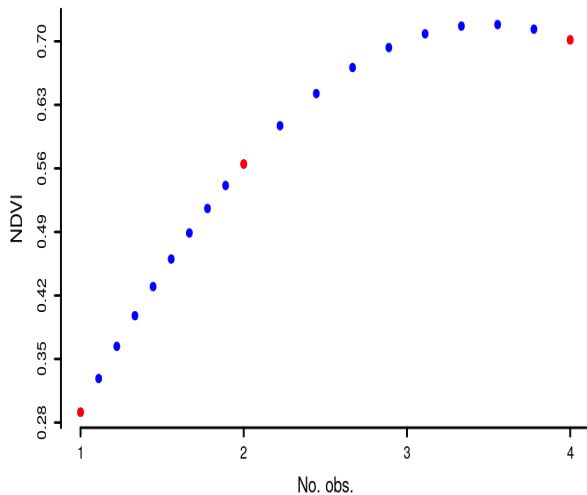
Interpolación Polinomial



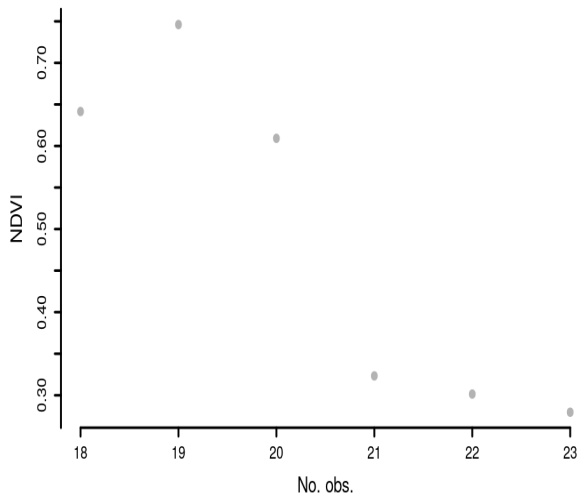
Interpolación Polinomial continuación



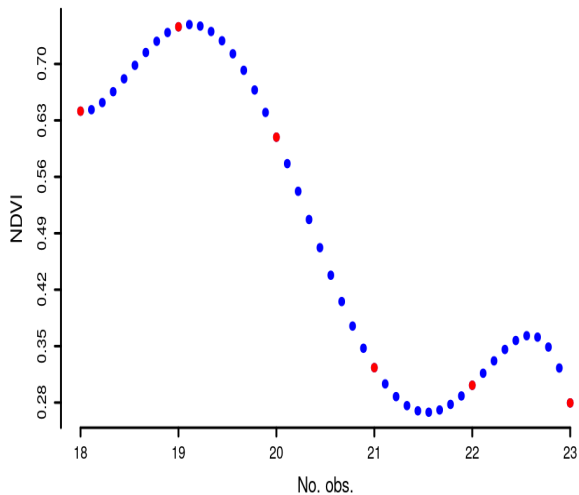
Interpolación Polinomial continuación



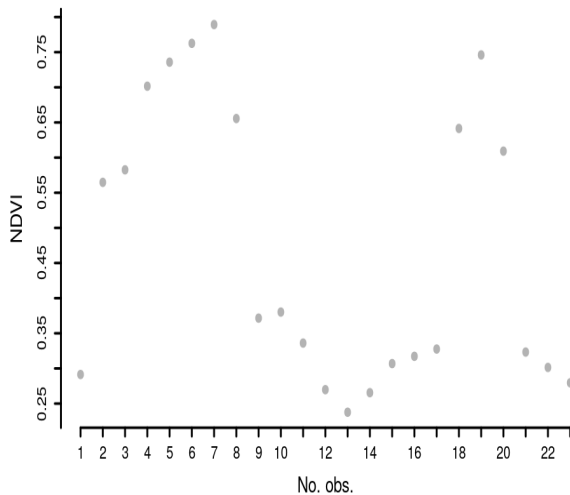
Interpolación Polinomial continuación



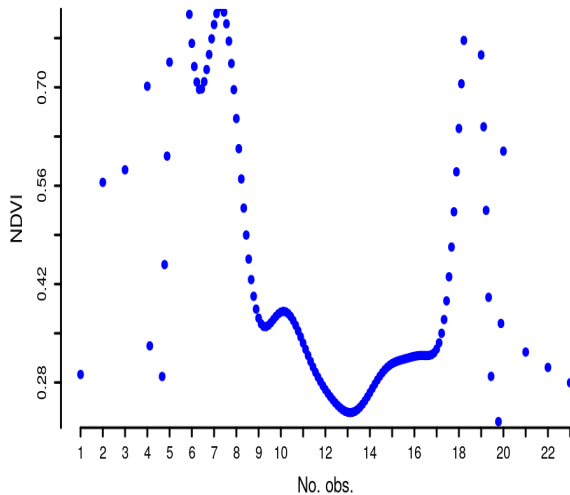
Interpolación Polinomial continuación



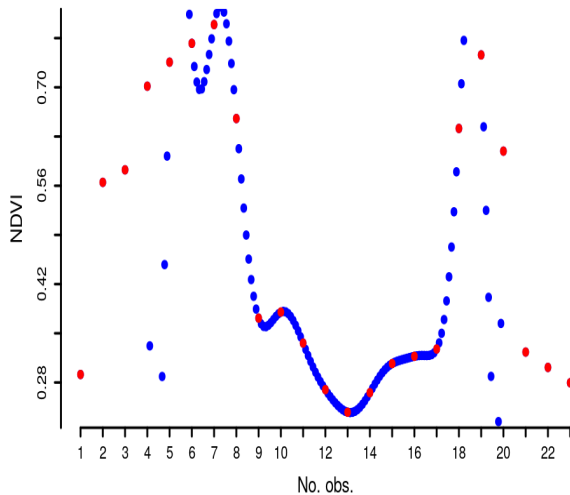
Interpolación Polinomial continuación



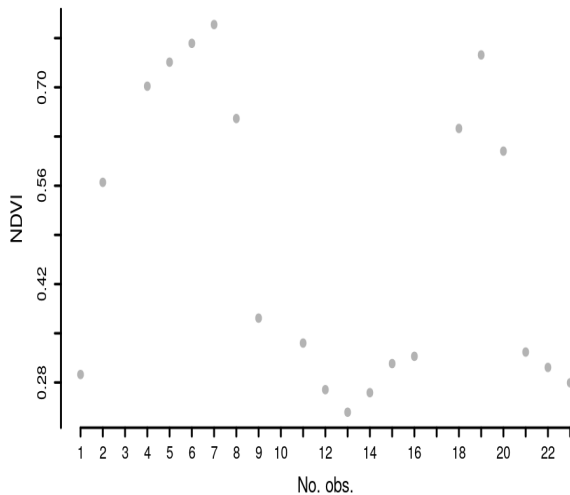
Interpolación Polinomial continuación



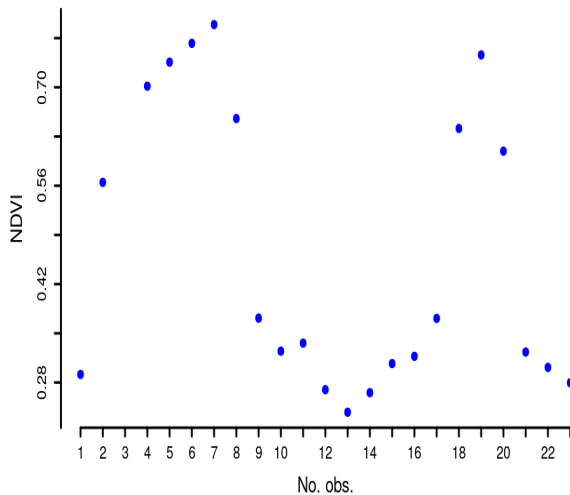
Interpolación Polinomial continuación



Interpolación Polinomial (missing values)



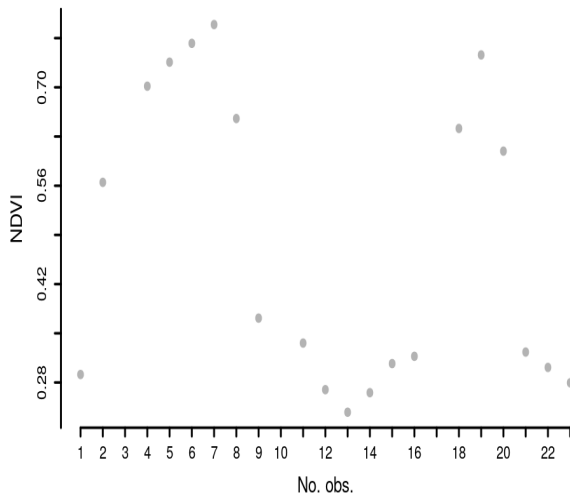
Interpolación Polinomial (missing values)



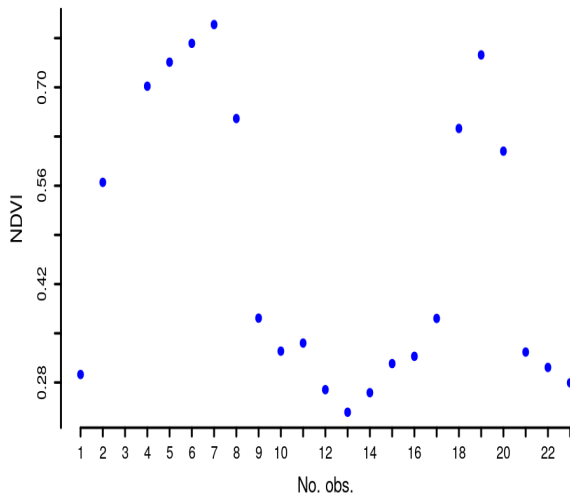

```
> yMask
[1] 0.291400 0.564800      NA 0.701600 0.735800 0.762600
     0.789300 0.655600 0.371600      NA 0.336100 0.269800
     0.237700 0.265500 0.306900 0.317250      NA 0.641475
[19] 0.746100 0.609200 0.323300 0.301450 0.279600
```

```
> LagrangePoly(x = 1:23, y = yMask)$interpol
[1] 0.2914000 0.5648000 14.1775204 0.7016000 0.7358000
     0.7626000 0.7893000 0.6556000 0.3716000 0.3246147
     0.3361000 0.2698000 0.2377000 0.2655000 0.3069000
[16] 0.3172500 0.3711811 0.6414750 0.7461000 0.6092000
     0.3233000 0.3014500 0.2796000
```

Interpolación Polinomial Baricéntrica



Interpolación Polinomial Baricéntrica



Representación polinomial alternativa

- Consideremos la pareja (x_i, y_i) y supongamos que existe un polinomio $p_n(\cdot)$ de grado n tal que $p_n(x_i) = y_i$. Entonces

$$y_i = p_n(x_i) = \beta_n x_i^n + \beta_{n-1} x_i^{n-1} + \cdots + \beta_1 x_i + \beta_0$$

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- Utilizando el mismo argumento para cada una de las parejas $(x_1, y_1), \dots, (x_n, y_n)$ podemos escribir

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

o matricialmente

$$\mathbf{X}\boldsymbol{\beta} = \mathbf{y}$$

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o matricialmente

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- Para encontrar los coeficientes $\boldsymbol{\beta}$, tenemos

$$\boldsymbol{\beta} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{y}$$

Representación polinomial alternativa

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$$y_i = p_n(x_i) = \beta_n x_i^n + \beta_{n-1} x_i^{n-1} + \cdots + \beta_1 x_i + \beta_0$$

- Utilizando el mismo argumento para cada una de las parejas $(x_1, y_1), \dots, (x_n, y_n)$ podemos escribir

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

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$$\mathbf{X}\boldsymbol{\beta} = \mathbf{y}$$

- Para encontrar los coeficientes $\boldsymbol{\beta}$, tenemos

$$\boldsymbol{\beta} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{y}$$

- Por cuestiones de **estabilidad**, podemos usar

$$\boldsymbol{\beta}_\lambda = (\mathbf{X}^\top \mathbf{X} + \lambda \mathbf{I}_n)^{-1} \mathbf{y},$$

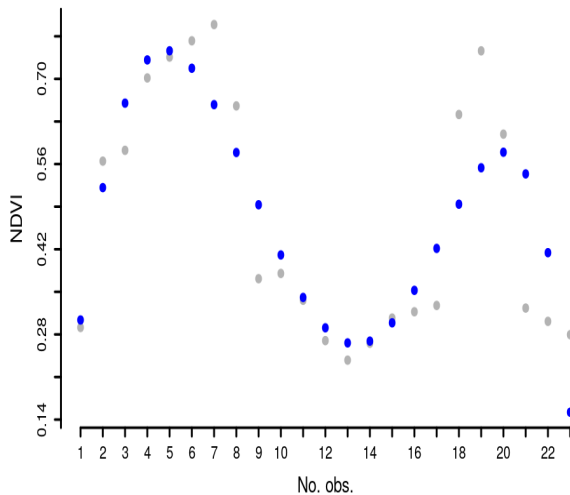
donde el valor de λ es pequeño.

Dificultades numéricas

```
> poly23rd <- getPolyInterpol(x = 1:23, y = data2, n = 23)
Error in solve.default(t(vanderMondeMatrix) %*% vanderMondeMatrix + 0.1 * :
  system is computationally singular: reciprocal condition number = 1.40968e-64

> getPolyInterpol(x = 1:23, y = data2, n = 6)
Error in solve.default(t(vanderMondeMatrix) %*% vanderMondeMatrix + 0.1 * :
  system is computationally singular: reciprocal condition number = 2.68731e-18
```

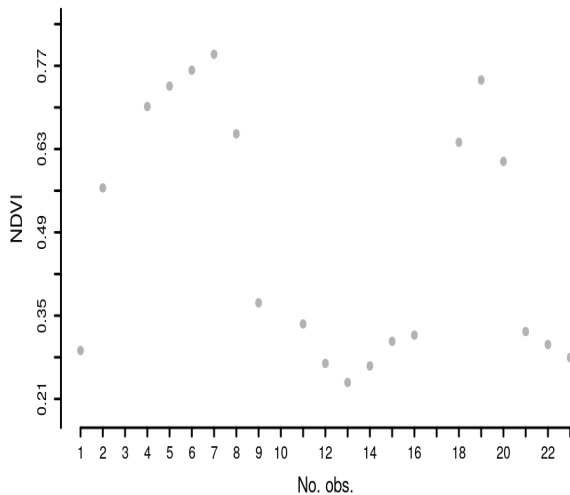

Aproximación con polinomio de quinto grado



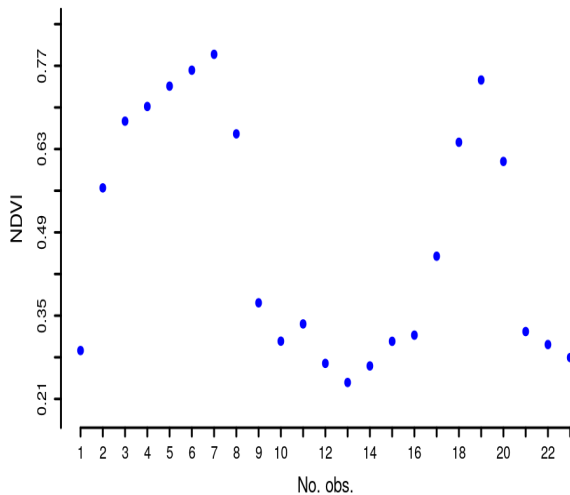
SPLINE



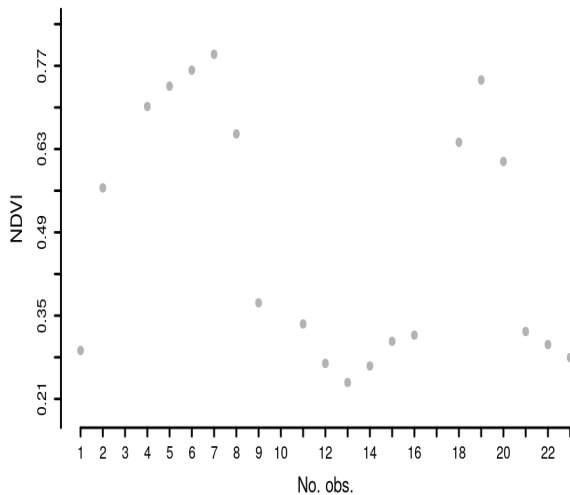
Interpolación vía spline natural cúbico



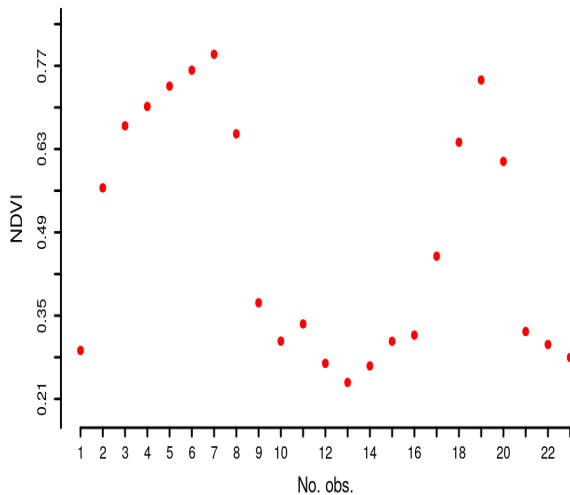
Interpolación vía spline natural cúbico



Interpolación vía spline monótono cúbico



Interpolación vía spline monótono cúbico



Mean Squared Error

- Estimador W_n , valor a estimar ϑ
- Conociendo la *distribución* de ϑ :

$$\text{MSE}[W_n] = E_{\vartheta}[(W_n - \vartheta)^2]$$

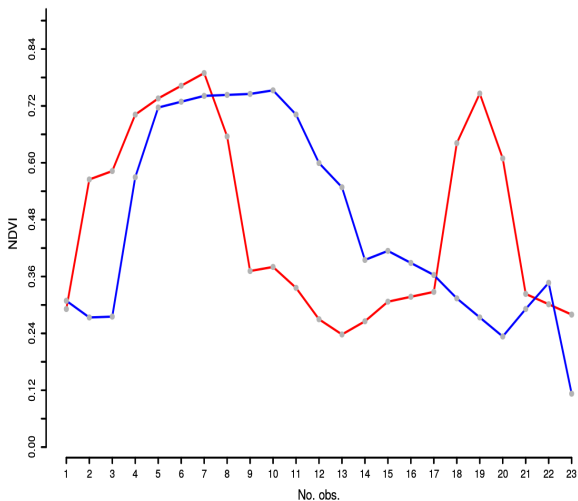
- Propiedad

$$\text{MSE}[W_n] = \left[\overbrace{E_{\vartheta}[W_n] - \vartheta}^{\text{SESGO}} \right]^2 + \text{VAR}(W_n)$$

- Si W_n y V_n son estimadores de ϑ , diremos que W_n es mejor que V_n (**MSE más pequeño**) si

$$\text{MSE}[W_n] < \text{MSE}[V_n].$$

Par de NDVIs



Setup 1: 3 missing values

	Rojo	Azul
Previous	0.002654478	0.001716745
Next	0.00591413	0.003776053
Mean	0.002179957	0.001125351
Linear	0.002016557	0.0009935881
Lagrange	23362601	308781446
Barycentric	23362495	308781442
Spline natural	0.001468945	0.0004662435
Spline monótono	0.001449537	0.0004731261

MSE de varios interpoladores aplicados a las trayectorias de NDVI mostradas previamente; se remueven 3 puntos en 5 escenarios distintos para **simular** missing values.

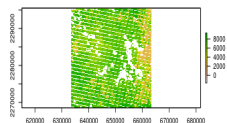
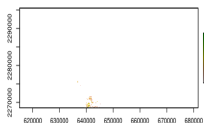
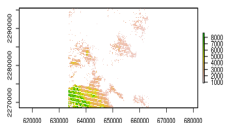
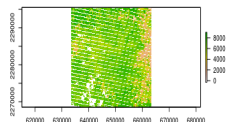
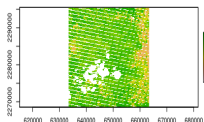
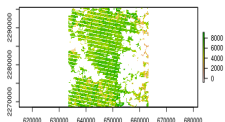
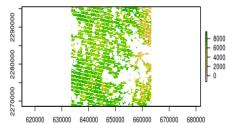
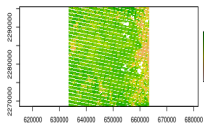
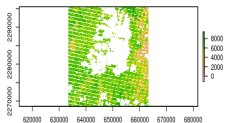
Setup 2: 11 missing values

	Rojo	Azul
Previous	0.0305365	0.01633331
Next	0.0195339	0.008417934
Mean	0.01255644	0.00637029
Linear	0.01067903	0.005935692
Lagrange	420610.2	49444.3
Barycentric	420610.2	49444.3
Spline natural	0.01097981	0.004674709
Spline monótono	0.01089073	0.004633711

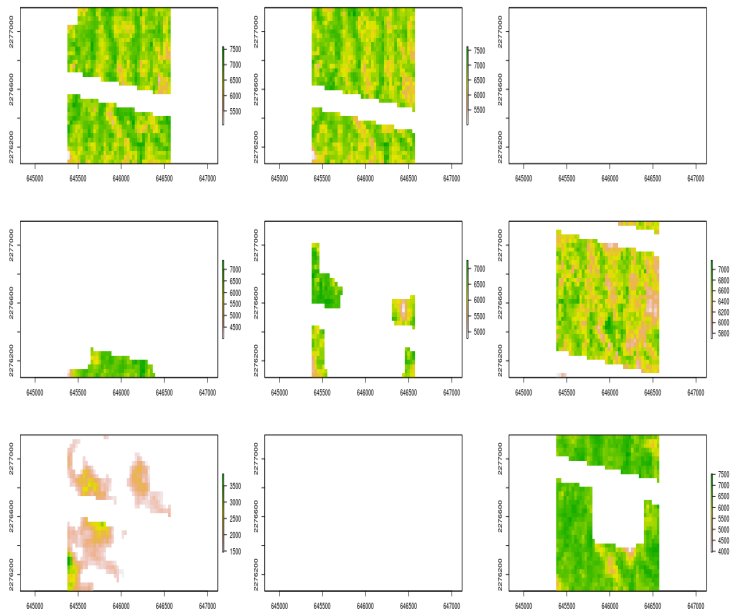
MSE de varios interpoladores aplicados a las trayectorias de NDVI mostradas previamente; se remueven 11 puntos en 5 escenarios distintos para **simular** missing values.

G A P F I L L

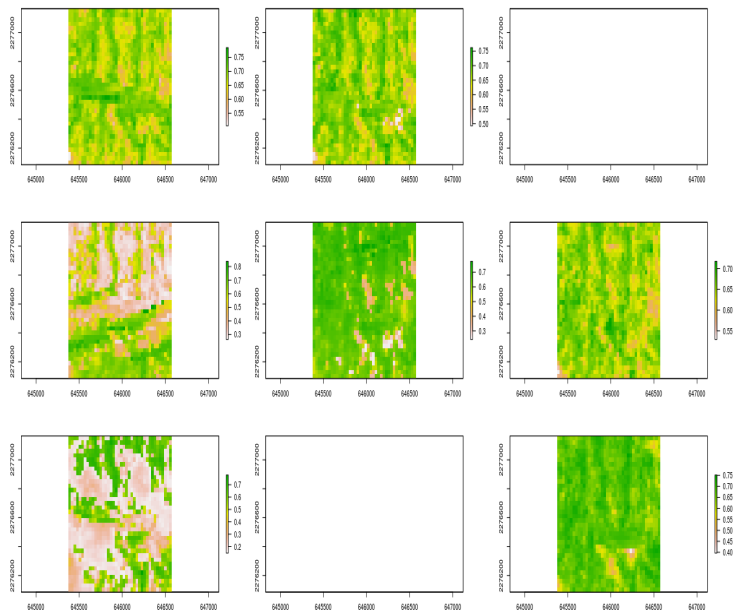
La Primavera



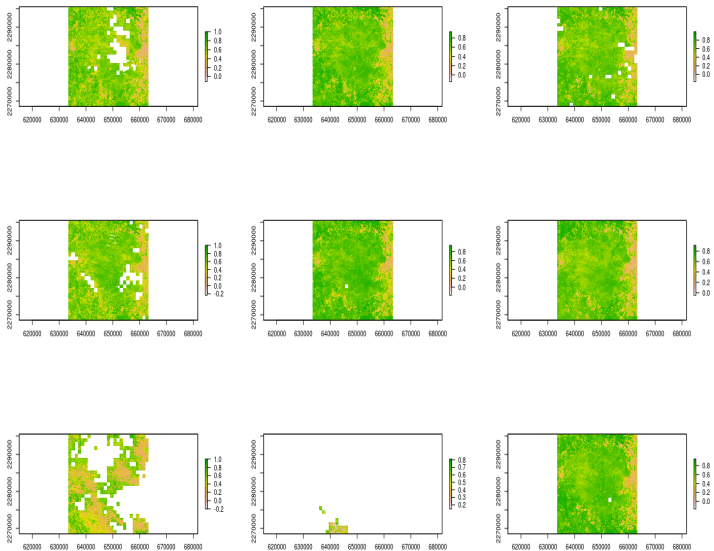
Recortes para aplicar gapfill



Aplicación de `gapfill`



La Primavera, después de `gapfill`



THAT'S ALL FOLKS!

- Colditz, R. R., Conrad, C., Wehrmann, T., Schmidt, M., and Dech, S. (2008). TiSeG: A flexible software tool for time-series generation of MODIS data utilizing the quality assessment science data set. *IEEE Trans. Geosci. Remote Sens.*, 46(10):3296–3308.
- Gerber, F., de Jong, R., Schaepman, M. E., Schaepman-Strub, G., and Furrer, R. (2018). Predicting missing values in spatio-temporal remote sensing data. *IEEE Transactions on Geoscience and Remote Sensing*, 56(5):2841–2853.